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ON OPTIMIZATION OF STOCHASTIC LINEAR SYSTEMS WITH TIME DELAY

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PART I

Separation Theorem in Linear Stochastic Systems with Time Delay

Separation Theorem in Linear Stochastic Systems with Time Delay†

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ABSTRACT

For linear stochastic systems with time delay, the optimal control is derived that minimizes the ensemble average of a quadratic (in states and control) performance measure. The optimal control obtained is functionally dependent upon the expected values of the state variables conditioned on the measurements. It is shown that the optimal control and estimation can be performed independently; i.e., the separation theorem holds for the class of problems considered. The optimal control is linearly dependent upon the best estimates which minimize the expected value of the estimation error squared.

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Introduction

The well-known separation theorem states that the combined problem of optimal control and estimation can, under certain conditions, be treated as two independent problems. It holds, for example, when the state transition in the plant and the observation equations are linear in the state variables containing additive white Gaussian noises and when the performance criterion is quadratic in state and control [1,4,5,6]. A more rigorous treatment of these conditions is presented in [5] for continuous-time stochastic systems. The separation theorem for discrete-time stochastic systems is given in [1]. The purpose here is to present the separation theorem for linear stochastic systems described by linear differential-difference equations of retarded type when the performance measure to be minimized is a quadratic function of the state variables and control.

A differential-difference equation is an equation which contains an unknown function and its derivatives which are evaluated at the values of the arguments differing by some specified amount. Such mathematical models appear commonly in aerospace application as well as in industrial processes. The state transition in these systems is a function, say, of state $x(t)$ evaluated at time t and state $x(t-\tau)$ evaluated at time $t-\tau$, where τ represents the time delay.

Necessary conditions to determine the optimal open-loop control for systems with time delay is fairly well established [e.g., 7]. More recently a feedback solution to the optimization of a system with linear plant-quadratic criterion has also been attained [2,8]. A solution to the optimal filtering in linear systems with time delays was proposed in [3] using the principle of orthogonal projections in Hilbert space. The optimal control is derived here for linear stochastic systems with time delay by means of the dynamic programming. Then, the known results of [3]

to the optimal filtering problem are applied to the case which results from the application of the optimal control in the stochastic system with time delay.

Statement of the Problem

The state transition of a plant is governed by a stochastic differential-difference equation

$$dx(t) = A_1(t)x(t)dt + A_2(t)x(t-\tau)dt + Bu(t)dt + D(t)\, dw(t) \tag{1}$$

where $t \in [t_o, T]$, $T < \infty$; $x(t) = \mathrm{col}[x_1(t), \ldots, x_n(t)]$ represents the system state at time t, and $x(t-\tau)$ at time $t-\tau$, where τ is a constant; $A_1(t)$, $A_2(t)$, $B(t)$ and $D(t)$ are bounded matrices with elements in C' for all t; $w(t)$ signifies a Brownian motion process with covariance (the superscript denotes transposition).

$$E\{[w(t_2) - w(t_1)]\, [w(t_2) - w(t_1)]'\} = Q_1(t_1)[t_2-t_1], \quad t_2 > t_1 \tag{2a}$$

$$E[w(t_2) - w(t_1)] = 0 \tag{2b}$$

The initial function represents a Gaussian process specified by

$$E[x(t)] = \bar{x}(t), \qquad t_o-\tau \leq t \leq t_o \tag{3a}$$

$$E\{[x(t_o+\theta) - \bar{x}(t_o+\theta)]\, [x(t_o+\sigma) - \bar{x}(t_o+\sigma)]'\} = P(t_o, \theta, \sigma) \tag{3b}$$

where $-\tau < \theta$, $\sigma \leq 0$.

The observations are performed according to

$$dz(t) = C_1(t)x(t)dt + C_2(t)x(t-\tau)dt + dv(t) \tag{4}$$

where $C_1(t)$ and $C_2(t)$ are bounded and continuous matrices of proper dimensions, $\forall\, t \in [t_o, T]$ and $v(t)$, $t \in [t_o, T]$ is a Brownian motion process with the covariance

$$E\{[v(t_1)-v(t_2)]\, [v(t_1)-v(t_2)]'\} = Q_2(t_1)[t_2-t_1], \quad t_2 > t_1 \tag{5a}$$

$$E[v(t_2)-v(t_1)] = 0 \tag{5b}$$

It is also assumed that $x(t)$, $t \in [t_o-\tau, t_o)$ and $w(t)$, $v(t)$, $t \in [t_o, T]$ are independent random processes.

The control problem is to determine the deterministic optimal control $u^*(t,\cdot)$ so as to minimize

$$J[u] = E_{z,t}\{\int_{t_0}^{T} [x'(t)W(t)x(t) + u'(t)R(t)u(t)]dt\} \qquad (6)$$

where $W(t)$ and $R(t)$ are continuous matrices which are positive semidefinite and positive definite, respectively, $\forall t$. The operator $E_{z,t}$ signifies a conditional expectation, i.e., $E_{z,t}\{\cdot\} = E\{\cdot|z(\xi),\ t_0 \leq \xi \leq t\}$.

Before solving the stochastic optimum control problem, some preliminary material is first presented.

Introductory Material

Suppose we are given the following differential equation of retarded type:

$$\dot{\bar{x}}(t) = F_1(t)\bar{x}(t)+F_2(t)\bar{x}(t-\tau) + \int_{-\tau}^{0} F_3(t,s)\bar{x}(t+s)ds \qquad (7)$$

where $t\epsilon[t_0,T]$; $F_1(t)$, $F_2(t)$ and $F_3(t,s)$ are bounded matrices with elements in C' for all t. $F_3(t,s)$ signifies the kernel of the equation. The initial function for system (7) is given by $\bar{x}(t)$, $t_0-\tau \leq t \leq t_0$.

Let $\Phi(t,s)$ represent an $(n \times n)$-matrix, which satisfies

$$\frac{\partial\Phi(t,s)}{\partial t} = F_1(t)\Phi(t,s)+F_2(t)\Phi(t-\tau,s)+\int_{-\tau}^{0} F_3(t,\alpha)\Phi(t+\alpha,s)d\alpha \qquad (8)$$

where $t\epsilon[t_0,T]$; $\Phi(t,t) = I$, $\Phi(t,s) = 0$ for $t < s$. It is noted that $\Phi(t,s)$ corresponds to the fundamental matrix of the ordinary differential equations. Also, $\Phi(s,t)$, $s \geq t$, satisfies the adjoint equations relative to the second argument:

$$\frac{\partial\Phi(s,t)}{\partial t} = -\Phi(s,t)F_1(t) - \Phi(s,t+\tau)F_2(t+\tau)$$
$$- \int_{-\tau}^{0}\Phi(s,t+\tau+\alpha)F_3(t+\tau+\alpha,-\alpha-\tau)d\alpha;\ t_0 \leq t \leq T-\tau \qquad (9)$$

$$\frac{\partial\psi(s,t)}{\partial t} = -\psi(s,t)F_1(t) - \int_{-\tau}^{0} \psi(s,t+\tau+\alpha)F_3(t+\tau+\alpha,-\alpha-\tau)d\alpha, \quad T-\tau \le t \le T \qquad (10)$$

where the boundary conditions are provided by $\psi(s,s) = I$ and $\psi(s,t) = 0$ for $s < t$.

The solution to equation (7) can now be written

$$\bar{x}(t) = \psi(t,t_o)\bar{x}(t_o) + \int_{-\tau}^{0} \tilde{\psi}(t,t_o+\tau+\sigma)\bar{x}(t_o+\sigma)d\sigma \qquad (11a)$$

where

$$\tilde{\psi}(t,t_o+\tau+\sigma) = \psi(t,t_o+\tau+\sigma)F_2(t_o+\tau+\sigma) + \int_{-\tau}^{\sigma} \psi(t,t_o+\tau+\alpha)F_3(t_o+\tau+\alpha,-\alpha-\tau+\sigma)d\alpha \qquad (11b)$$

The solution to the process $x(t)$ in equation (1) is defined as

$$x(t) = \psi(t,t_o)x(t_o) + \int_{-\tau}^{0} \tilde{\psi}(t,t_o+\tau+\sigma)x(t_o+\sigma)d\sigma + $$
$$+ \int_{t_o}^{t} \psi(t,\sigma)B(\sigma)u(\sigma)d\sigma + \int_{t_o}^{t} \psi(t,\sigma)D(\sigma)dw(\sigma) \qquad (12)$$

where $\psi(t,\cdot)$ is determined by equation (8) with $F_3(t,s) \equiv \underline{0}$ (null-matrix). One observes that the process $x(t)$ is Gaussian since $x(\xi)$, $t_o-\tau \le \xi \le t_o$, and $\{dw(t)\}$ are Gaussian[4].

Having presented the preliminary material, the optimization problem can now be solved by the method of the dynamic programming.

Solution to the Stochastic Control Problem

The optimal control to the stochastic system with the time delay is determined by the dynamic programming method. As usual, one assumes that the system starts evolving at time $t\in[t_o,T]$ from state $x(\xi)$, $t-\tau \le \xi \le t$. The minimum value of the functional specified in equation (6) is denoted by $V[\bar{x}_t,t]$, i.e.,

$$V[\bar{x}_t,t] = \text{Min}\ E_{z,t}\ \{ \int_t^T [\|x(s)\|^2_{W(s)} + \|u(s)\|^2_{R(s)}]ds \} \qquad (13)$$

where $\|x(s)\|^2_{W(s)} = x'(s)W(s)x(s)$.

The first term on the right of equation (13) can be expressed as follows

$$E_{z,t}\left\{\int_t^T \|x(s)\|^2_W\, ds\right\} = \int_t^T \|\bar{x}(s)\|^2_{W(s)} ds + \int_t^T \mathrm{tr}[W(s)P(s,0,0)]ds \quad (14)$$

where $\bar{x}(s) = E_{z,t}\{x(s)\}$ (15a)

$$P(s,0,0) = E_{z,t}\{[x(s) - \bar{x}(s)][x(s) - \bar{x}(s)]'\} \quad (15b)$$

If the states of a system evolve in time according to equation (7), equation (11a) can be substituted for $\bar{x}(s)$ in equation (14) (naturally, after changing t to s and t_o to t in equation (11a)). Then, we can write

$$\int_t^T \|\bar{x}(s)\|^2_{W(s)} ds = \bar{x}'(t)\bar{K}_o(t)\bar{x}(t) + \bar{x}'(t)\int_{-\tau}^0 \bar{K}_1(t,\sigma)\bar{x}(t+\sigma)d\sigma +$$

$$\left(\int_{-\tau}^0 \bar{x}'(t+\sigma)\bar{K}_1'(t,\sigma)d\sigma\right)\bar{x}(t) + \int_{-\tau}^0\int_{-\tau}^0 \bar{x}'(t+\sigma)\bar{K}_2(t,\sigma,\alpha)\bar{x}(t+\alpha)d\sigma d\alpha \quad (16)$$

where

$$\bar{K}_o(t) = \int_t^T \Psi'(s,t)W(s)\Psi(s,t)ds\ , \qquad t_o \leq t \leq T \quad (17)$$

$$\bar{K}_1(t,\sigma) = \int_t^T \Psi'(s,t)W(s)\tilde{\Psi}(s,t+\tau+\sigma)ds;\ -\tau < \sigma \leq 0\ ,\ -\tau < \alpha \leq 0 \quad (18)$$

$$\bar{K}_2'(t,\alpha,\sigma) = \bar{K}_2(t,\sigma,\alpha) = \int_t^T \tilde{\Psi}'(s,t+\tau+\sigma)W(s)\tilde{\Psi}(s,t+\tau+\alpha)ds \quad (19)$$

One observes that equations (16), (17) and (18) at time $t = T$ become:

$$\bar{K}_o(T) = \underline{0},\quad \bar{K}_1(T,\sigma) = \underline{0},\quad \bar{K}_2(T,\alpha,\sigma) = \underline{0} \quad (20)$$

for $-\tau < \sigma \leq 0$ and $-\tau < \alpha \leq 0$.

Suppose now that $x(t)$ evolves in time according to equation (1), where $u(t) = u[\bar{x}_t,t]$ is specified by

$$u[\bar{x}_t,t] = -R^{-1}B'K_o(t)\bar{x}(t) - R^{-1}B'\int_{-\tau}^0 K_1(t,\sigma)\bar{x}(t+\sigma)d\sigma \quad (21)$$

where the continuous matrices $K_o(t)$ and $K_1(t,\sigma)$ are to be determined. Then, $\bar{x}(t)$ is governed by

$$\dot{\bar{x}}(t) = A_1(t)\bar{x}(t) + A_2(t)\bar{x}(t-\tau) + B(t)u[\bar{x}_t, t] \tag{22}$$

Now it follows that the performance criterion can be expressed as (see appendix)

$$E_{z,t}\left\{\int_t^T \left[\|x(s)\|^2_{W(s)} + \|u(s)\|^2_{R(s)}\right] ds\right\}$$

$$= \int_t^T \left\{\|\bar{x}(s)\|^2_{W(s)} + tr\left[W(s)P(s,0,0)\right] + \|u[\bar{x}_s, s]\|^2_{R(s)}\right\} ds$$

$$= \|\bar{x}(t)\|^2_{K_o(t)} + \int_{-\tau}^{0} \left[\bar{x}'(t)K_1(t,\sigma)\bar{x}(t+\sigma) + \bar{x}'(t+\sigma)K_1'(t,\sigma)\bar{x}(t)\right] d\sigma$$

$$+ \int_{-\tau}^{0}\int_{-\tau}^{0} \bar{x}'(t+\sigma)K_2(t,\sigma,\alpha)\bar{x}(t+\alpha)\, d\sigma d\alpha + S(t) \tag{23}$$

where $K_o(t)$, $K_1(t,\sigma)$ and $K_2(t,\sigma,\alpha)$ are continuous (nxn)-matrices, and $S(t)$ is a continuous scalar function dependent upon the covariance of the plant noise. The functional equations (given by (A6), (A7) and (A8) in the appendix) can be shown to be equivalent to the following set of partial differential equations:

$$\frac{dK_o(t)}{dt} = - A_1'(t)K_o(t) - K_o(t)A_1(t) + K_o(t)BR^{-1}B'K_o(t) - K_1(t,0) - K_1{}'(t,0) - W(t) \tag{24}$$

$$\frac{\partial K_1(t,\sigma)}{\partial t} - \frac{\partial K_1(t,\sigma)}{\partial \sigma} = - A_1'(t)K_1(t,\sigma) + K_o(t)BR^{-1}BK_1(t,\sigma) - K_2(t,0,\sigma) \tag{25}$$

$$\frac{\partial K_2(t,\alpha,\sigma)}{\partial t} - \frac{\partial K_2(t,\alpha,\sigma)}{\partial \sigma} - \frac{\partial K_2(t,\alpha,\sigma)}{\partial \alpha} = K_1'(t,\alpha)BR^{-1}B'K_1(t,\sigma) \tag{26}$$

Equations (24), (25) and (26) can be derived (after some tedious manipulations) by substituting equation (12) into equation (14), collecting the terms for $K_o(\cdot)$, $K_1(\cdot,\cdot)$ and $K_2(\cdot,\cdot,\cdot)$ in the resulting equation, differentiating their expressions and making use of equations (21) and (22). A more detailed discussion is given in the appendix. A more direct derivation of equations (24), (25) and (26) can be based on equation (29), which is obtained in the sequel.

The boundary conditions associated with equations (24), (25) and (26) are:

$$K_o(T) = \underline{0}\ ; \qquad K_1(T,\sigma) = \underline{0}\ ; \qquad K_2(T,\alpha,\sigma) = \underline{0}\ ; \qquad K_o(t)A_2(t) = K_1(t,-\tau)\ ;$$
$$K_2(t,-\tau,\sigma) = A_2'(t)K_1(t,\sigma)\ ; \qquad K_2(t,\alpha,\sigma) = K_2'(t,\sigma,\alpha) \tag{27}$$

Equation (27) can be verified directly from the defining expressions of $K_o(t)$, $K_1(t,\sigma)$ and $K_2(t,\alpha,\sigma)$.

Now it can be seen that $u(\bar{x}_t,t)$ in equation (21) is the optimal control [4]. Namely, one obtains for $t=t_o$ by writing $L(t,x,u)$ for the integrand in equation (6) and observing $V[\bar{x}_T,T] = 0$:

$$V[\bar{x}_t^*,t_o] = -E_{z,t}\int_{t_o}^{T} \frac{dV[\bar{x}_t^*,t]}{dt}\,dt$$
$$\leq -E_{z,t}\int_{t_o}^{T} \left\{\frac{dV[\bar{x}_t^*,t]}{dt} + L(t,x^*,u^*) - L(t,x,u)\right\} dt$$
$$= E_{z,t}\int_{t_o}^{T} L(t,x,u)\,dt \tag{28}$$

Hence, a sufficient condition for the optimality is determined by solving

$$\min_u \left\{ \frac{dV[\bar{x}_t, t]}{dt} + L(t,\bar{x},u) \right\} = 0 \tag{29}$$

where the expression for $V[\cdot,\cdot]$ is specified by equation (23). (In equation (23), $S(t)$ depends implicitly on the plant covariance but does not depend upon $u(\cdot,\cdot)$). Straight forward calculations show that if the control in equation (21) is chosen, equations (28) and (29) are fulfilled. Thus, the optimal control $u^*(\bar{x}_t, t)$ is given by equation (21). It indicates that (i) the optimal control is generated linearly by the expected values of the state variables conditioned on the measurements; (ii) the feedback gains are independent of the observations and can be computed in advance ("off-line"). The estimates of the state variables needed are the optimal estimates that minimize the conditional mean of the squared estimation error [3].

Since the optimal control $u^*[\bar{x}_t, t]$ given by equation (21) is deterministic, the optimal control and the optimal filtering can be solved independently. Thus the separation theorem for stochastic linear systems with time delay is established.

Separation Theorem: For stochastic linear systems with time delay described by equation (1) through (5), the optimal control that minimizes performance criterion (5) is specified by equation (21). The optimal feedback gains and the optimal estimates $\bar{x}(t+\sigma) = E\{x(t+\sigma)|z(\beta),\ t_o \le \beta \le t,\ t-\tau < t+\sigma \le t\}$ can be determined independently.

One observes that the feedback gains in equation (21) are deterministic, and they do not depend upon the statistics of the noises. Moreover, the optimal control (21) is the same as the optimal control of a deterministic system obtained by replacing the random variables in equations (1), (4) and (6) by their average values; i.e., the certainty-equivalence is valid for the class of problems considered here.

An alternative formal derivation of the dynamic programming equation is presented in Appendix B by using the principle of the optimality.

Optimal Estimation

In order to apply the optimal control $u^*(\bar{x}_t,t)$ in equation (21), the optimal estimates $\bar{x}(\xi) = E_{\pi,t}\{x(\xi)\}$ for $t-\tau \le \xi \le t$ (the conditional mean value of $x(\xi)$) must be generated. When the system moves under the influence of $u^*(\bar{x}_t,t)$, the plant equation is

$$dx(t) = \Big\{A_1(t)x(t) + A_2(t)x(t-\tau) - BR^{-1}B'\Big[K_0(t)\bar{x}(t) + \int_{-\tau}^{0} K_1(t,\sigma)\bar{x}(t+\sigma)d\sigma\Big]\Big\} dt + D(t)\, dw(t) \tag{30}$$

and the observations are performed according to equation (4).

The optimal estimation equations can be written by applying the results of [3]. The estimates of the state variables that minimize

$$E\left\{\left[\bar{x}(t+\theta) - x(t+\theta)\right]'\left[\bar{x}(t+\theta) - x(t+\theta)\right] \big| z(\xi),\ t_0 \le \xi \le t\right\}; -\tau < \theta \le 0 \tag{31}$$

are determined by equations(32) and (33).(formally)

$$\frac{\partial \hat{x}(t)}{\partial t} = A_1(t)\bar{x}(t) + A_2(t)\bar{x}(t-\tau) - BR^{-1}B'\Big[K_0(t)\bar{x}(t) + \int_{-\tau}^{0} K_1(t,\sigma)\bar{x}(t+\sigma)d\sigma\Big] + - G^0(t.0,t)\left[\dot{z}(t) - C_1(t)\bar{x}(t) - C_2(t)\hat{x}(t-\tau)\right];\ \theta = 0 \tag{32}$$

$$\frac{\partial \hat{x}(t+\theta)}{\partial t} + \frac{\partial \hat{x}(t+\theta)}{\partial \theta} = G^0(t,\theta,t)\left[\dot{z}(t) - C_1(t)\hat{x}(t) - C_2(t)\hat{x}(t-\tau)\right] \quad -\tau < \theta \le 0 \tag{33}$$

The gain of the estimator is specified by

$$G^0(t,\theta,t) = \left[P(t,\theta,0)\, C_1'(t) + P(t,\theta,-\tau)\, C_2'(t)\right] Q_2^{-1}(t) \tag{34}$$

where $P(\cdot,\cdot,\cdot)$ signifies the error covariance

$$P(t,\theta,\sigma) = E_{z,t}\left\{\left[\hat{x}(t+\theta)-x(t+\theta)\right]\left[\hat{x}(t+\sigma)-x(t+\sigma)\right]'\right\}, -d \leq \theta,\ \sigma \leq 0 \tag{35}$$

This covariance is governed by the following set of partial differential equations

$$\begin{aligned}\frac{\partial P(t,0,0)}{\partial t} = {} & A_1(t)P(t,0,0) + A_2(t)P(t,\tau,0) + P(t,0,0)A_1'(t) \\ & + P'(t,0,\tau)A_2'(t) + Q_1(t) - P(t,0,0)C_1'(t)Q_2^{-1}(t)C_1(t)P(t,0,0) \\ & - P(t,0,0)C_1'(t)Q_2^{-1}(t)C_2P(t,\tau,0) - P(t,0,\tau)C_2'(t)Q_2^{-1}(t)C_1(t)P(t,0,0) \\ & - P(t,0,\tau)C_2'(t)Q_2^{-1}(t)C_2(t)P(t,\tau,0)\end{aligned} \tag{36}$$

$$\begin{aligned}\frac{\partial P(t,0,\sigma)}{\partial t} + \frac{\partial P(t,0,\sigma)}{\partial \sigma} = {} & A_1(t)P(t,0,\sigma) + A_2(t)P(t,\tau,\sigma) \\ & -P(t,0,0)C_1'(t)Q_2^{-1}(t)C_1(t)P(t,0,\sigma)-P(t,0,0)C_1'(t)Q_2^{-1}C_2(t)P(t,\tau,\sigma) \\ & -P(t,0,\tau)C_2'(t)Q_2^{-1}(t)C_1(t)P(t,0,\sigma)-P(t,0,\tau)C_2'(t)Q_2^{-1}(t)\cdot \\ & C_2(t)P(t,\tau,\sigma)\end{aligned} \tag{37}$$

$$\begin{aligned}\frac{\partial P(t,\theta,\sigma)}{\partial t} + \frac{\partial P(t,\theta,\sigma)}{\partial \theta} + \frac{\partial P(t,\theta,\sigma)}{\partial \sigma} = {} & -P(t,\theta,0)C_1'(t)Q_2^{-1}C_1(t)P(t,0,\sigma) \\ & -P(t,\theta,0)C_1'(t)Q_2^{-1}C_2(t)P(t,\tau,\sigma)-P(t,\theta,\tau)C_2'(t)Q_2^{-1}C_1(t)P(t,0,\sigma) \\ & -P(t,\theta,\tau)C_2'(t)Q_2^{-1}C_2(t)P(t,\tau,\sigma)\end{aligned} \tag{38}$$

Equations (32) through (38) establish the solution to the optimal estimation. Equations (36) through (38) are independent of the observations, and can be computed "off-line". The computational difficulties involved are presently being explored.

The equations for the solution to the combined problem of the optimal control and estimation is now furnished. The block diagram in figure 1 displays the optimal system. The main difficulty in the "on-line" implementation is due to the realization of the term $-BR^{-1}B'(t)\int_{-\tau}^{0} K_1(t,\sigma)\bar{x}(t+\sigma)d\sigma$, which requires the solution of the optimal smoothing as well. As a first approximation, the integral can be replaced by a finite sum. Then the designer can use a finite number of controllers to operate on the optimal smoothed estimates, which can be computed (by means of a fixed-lag smoothing procedure). The problem of implementing the optimal solution for time delay systems is currently being investigated.

Conclusions

The optimal feedback control is determined for stochastic linear systems with time-invariant time delay so that the average value of a quadratic cost functional is minimized. The optimal control depends linearly on the expected values of the state variables conditioned on the measurements. They are the best estimates of the states which result in the minimum of the estimation error. The optimal feedback control and the optimal estimates can be determined independently.

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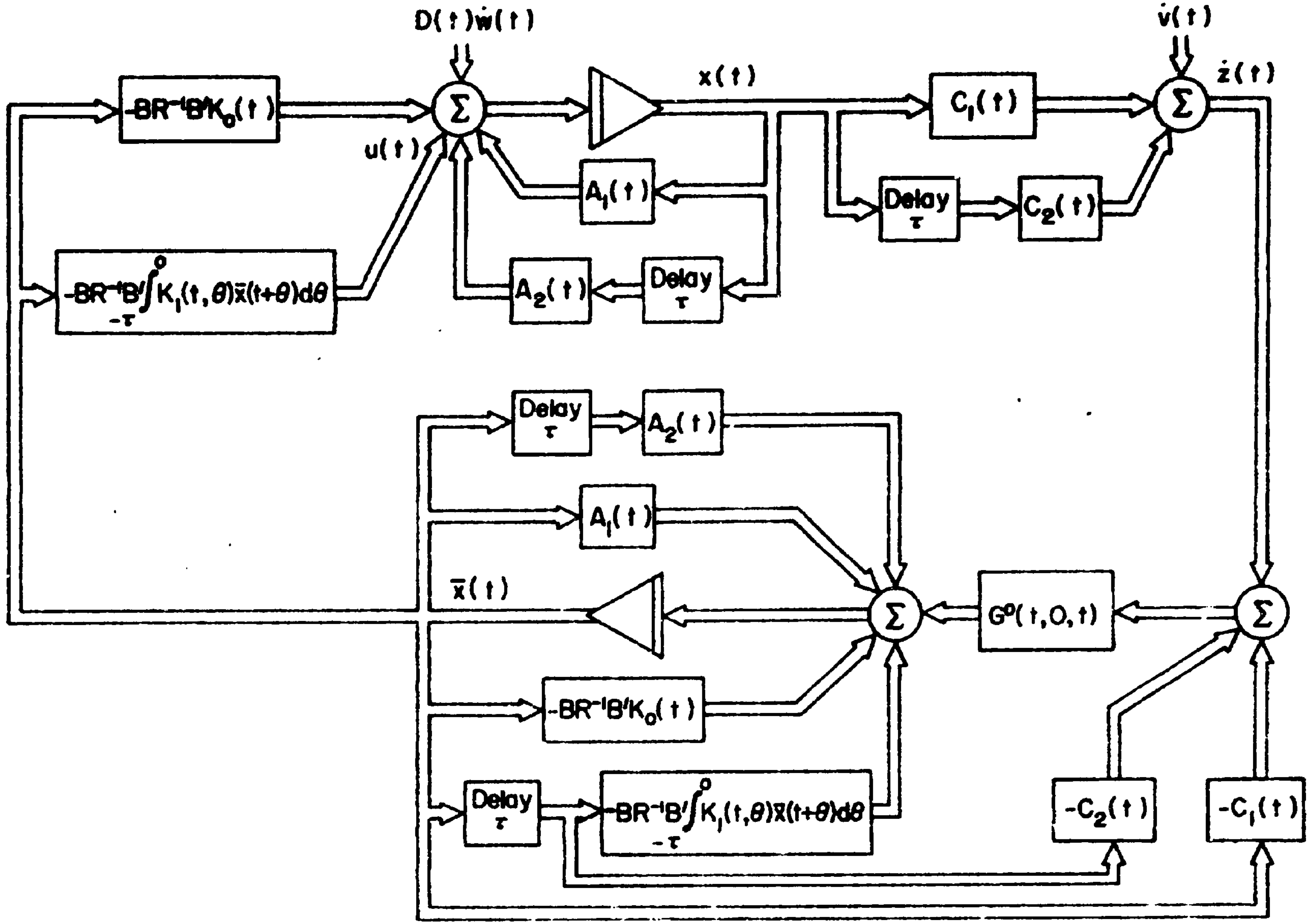
Figure 1.



D(t)ẇ(t)
x(t)
v̇(t)
ż(t)
-BR⁻¹B′K₀(t)
u(t)
-BR⁻¹B′∫K₁(t,θ)x̄(t+θ)dθ
A₁(t)
A₂(t)
Delay τ
C₁(t)
C₂(t)
x̄(t)
G⁰(t,0,t)
-C₂(t)
-C₁(t)

Appendix A

Derivation of Equations (23) through (26)

Suppose that systems (1) and (22) move under the influence of the control

$$u[\bar{x}_t, t] = -R^{-1}B'K_0(t)\bar{x}(t) - R^{-1}B'\int_{-\tau}^{0} K_1(t,\sigma)\,\bar{x}(t+\sigma)\,d\sigma \tag{A1}$$

where $K_0(t)$ and $K_1(t,\sigma)$ are gain matrices to be determined. The task is now to express $V[\bar{x}_t, t]$ in equation (13) in terms of $x(\xi)$, $t-\tau \leq \xi \leq t$, the initial function. One observes first that the solution, $\bar{x}(s)$ at time $s \geq t$, to equation (22) can be written by means of equations (11a) and (11b).

$$\bar{x}(s) = \Phi(s,t)\bar{x}(t) + \int_{-\tau}^{0} \tilde{\Phi}(s, t+\tau+\sigma)\bar{x}(t+\sigma)\,d\sigma \tag{A2}$$

when $\tilde{\Phi}(\cdot,\cdot)$ is specified by

$$\tilde{\Phi}(s, t+\tau+\sigma) = \Phi(s, t+\tau+\sigma)A_2(t+\tau+\sigma) - \int_{-\tau-\sigma}^{0} \Phi(s, t+\tau+\sigma+\alpha)BR^{-1}B'(t+\tau+\sigma+\alpha)\cdot$$

$$K_1(t+\tau+\sigma+\alpha; -\alpha-\tau)\,d\alpha \tag{A3}$$

Before substituting $\bar{x}(s)$ into equation (13), relation (14) is used in equation (13). The resulting equation can be rewritten by means of equation (A2):

$$\begin{aligned} V[\bar{x}_t, t] = \int_t^T \Big\{ &\bar{x}'(s)\left[W(s) + K_0(s)BR^{-1}B'(s)K_0(s)\right]\bar{x}(s) + \\ &\bar{x}'(s)K_0(s)BR^{-1}B'(s)\int_{-\tau}^{0} K_1(s,\sigma)\bar{x}(s+\sigma)\,d\sigma + \\ &\int_{-\tau}^{0} \bar{x}'(s+\sigma)K_1'(s,\sigma)d\sigma\; BR^{-1}B'(s)K_0(s)\bar{x}(s) + \\ &\int_{-\tau}^{0}\int_{-\tau}^{0} \bar{x}'(s+\sigma)K_1'(s,\sigma)BR^{-1}B'(s)K_1(s,\alpha)\bar{x}(s+\alpha)\,d\sigma\,d\alpha + \mathrm{tr}[W(s)P(s,0,0)]\Big\}\,ds \end{aligned} \tag{A4}$$

Now equation (A4) can be expressed solely in terms of $x(t+\xi)$, $-\tau < \xi \le 0$, by substituting equation (A2) into (A4). It results in

$$V[\bar{x}_t, t] = \bar{x}'(t)K_0(t)\bar{x}(t) + \int_{-\tau}^{0}\left[\bar{x}'(t)K_1(t,\sigma)\bar{x}(t+\sigma) + \bar{x}'(t+\sigma)K_1'(t,\sigma)\bar{x}(t)\right]d\sigma$$
$$+ \int_{-\tau}^{0}\int_{-\tau}^{0} \bar{x}'(t+\sigma)K_2(t,\sigma,\alpha)x(t+\alpha)\,d\sigma\,d\alpha \qquad \text{(A5)}$$

where $K_0(t)$, $K_1(t,\sigma)$ and $K_2(t,\sigma,\alpha)$ are specified by

$$K_0(t) = \int_t^T ds \left\{\psi'(s,t)\left[W(s)+K_0(s)BR^{-1}B'(s)K_0(s)\right]\psi(s,t)\right.$$
$$+ \psi'(s,t)K_0(s)BR^{-1}B'(s)\int_{-\tau}^{0} K_1(s,\alpha)\,\psi(s+\alpha,t)\,d\alpha$$
$$+ \int_{-\tau}^{0} \psi'(s+\alpha,t)K_1'(s,\alpha)BR^{-1}B'(s)K_0(s)\,\psi(s,t)\,d\alpha$$
$$\left.+ \int_{-\tau}^{0}\int_{-\tau}^{0} \psi'(s+\sigma,t)K_1'(s,\sigma)BR^{-1}B'(s)K_1(s,\alpha)\psi(s+\alpha,t)\,d\alpha\,d\sigma\right\} \qquad \text{(A6)}$$

$$K_1(t,\sigma) = \int_t^T ds \left\{\psi'(s,t)\left[W(s)+K_0(s)BR^{-1}B'(s)K_0(s)\right]\tilde{\psi}(s,t+\tau+\sigma) + \psi'(s,t)K_0(s)\cdot\right.$$
$$BR^{-1}B'(s)\int_{-\tau}^{0} K_1(s,\beta)\,\tilde{\psi}(s+\beta,t+\tau+\sigma)\,d\beta + \int_{-\tau}^{0} \psi'(s+\xi,t)K_1'(s,\xi)\,d\xi\,\cdot$$
$$BR^{-1}B'(s)K_0(s)\,\tilde{\psi}(s,t+\tau+\sigma) + \int_{-\tau}^{0}\int_{-\tau}^{0} \psi'(s+\xi,t)K_1'(s,\xi)\,\cdot$$
$$\left.BR^{-1}B'(s)K_1(s,\alpha)\,\tilde{\psi}(s+\alpha,t+\tau+\sigma)\,d\xi\,d\alpha\right\} \qquad \text{(A7)}$$

$$K_2(t,\sigma,\alpha) = \int_t^T ds \left\{\tilde{\psi}'(s,t+\tau+\sigma)\left[W(s)+K_0(s)BR^{-1}B'(s)K_0(s)\right]\tilde{\psi}(s,t+\tau+\alpha) + \right.$$
$$\tilde{\psi}'(s,t+\tau+\sigma)K_0(s)BR^{-1}B'(s)\int_{-\tau}^{0} K_1(s,\beta)\tilde{\psi}(s+\beta,t+\tau+\alpha)\,d\beta +$$
$$\int_{-\tau}^{0}\tilde{\psi}'(s+\beta,t+\tau+\sigma)K_1'(s,\beta)\,d\beta\; BR^{-1}B'(s)K_0(s)\tilde{\psi}(s,t+\tau+\alpha) +$$

$$+\int_{-\tau}^{0}\int_{-\tau}^{0}\tilde{\Phi}'(s+\xi,t+\tau+\sigma)K_1'(s,\xi)BR^{-1}B'(s)K_1(s,\beta)\tilde{\Phi}(s+\beta,t+\tau+\alpha)d\xi d\beta\Big\} \tag{A8}$$

where $\tilde{\Phi}(s,t+\tau+\sigma)$ is defined by equation (A3), and $\Phi(s,t)$ satisifes equations (9) and (10) with respect to t. Moreover, $S(t) = \mathrm{tr}\int_t^T W(s)P(s,0,0)ds$.

Equations (A6), (A7) and (A8) yield directly that

$$K_1(t,-\tau) = K_0(t)A_2(t) \tag{A9}$$

$$K_2(t,\sigma,\alpha) = K_2'(t,\alpha,\sigma) \qquad ; -\tau < \sigma \leq 0 \tag{A10}$$

$$K_2(t,\sigma,-\tau) = K_1'(t,\sigma)A_2(t) \quad ; -\tau < \alpha \leq 0 \tag{A11}$$

$$K_0(T) = \underline{0} \quad \text{(null-matrix)} \tag{A12}$$

$$K_1(T,\sigma) = \underline{0} \tag{A13}$$

$$K_2(T,\sigma,\alpha) = \underline{0} \tag{A14}$$

Equations (A9) through (A14) establish equations given by (27).

If equation (A6) is differentiated with respect to t, one can obtain equation (24) by making use of equations (9), (10) and equations (A6) and (A7). Moreover, if equation (A7) is used to generate $\partial K_1/\partial t - \partial K_1/\partial\sigma$, one obtains equation (25) by means of equations (9), (10) and (A8). Similarly, equation (A8) yields equation (26) after some tedious manipulations.

It is noted that equations (24) through (27) can also be obtained directly by means of equation (29).

Appendix B

Formal Derivation of Equation (29)

A formal derivation for the optimality condition (29) is presented here. To apply the dynamic programming, let the minimum value of the performance index (PI) for the process starting at time t from state $x(\xi)$, $t-d \leq \xi \leq t$ be denoted by

$$V\left[\bar{x}_t, t\right] = \underset{u}{\mathrm{Min}}\ E_{z,t}\left\{ \int_t^T \left[\|x(t)\|^2_{W(t)} + \|u(t)\|^2_{R(t)}\right] dt \right\}$$

$$= \underset{u}{\mathrm{Min}}\ E_{z,t}\left\{ \int_t^T L[t,x,u]\ dt\right\} \tag{B1}$$

where $\|\cdot\|$ signifies a Euclidean norm; x_t emphasizes the functional dependence of V on x; and $E_{z,t}\{\cdot\} = E\{\cdot|z(t),t\}$ represents a conditional expectation operation.

The application of the principle of optimality leads to

$$V\left[\bar{x}_t, t\right] = \underset{u}{\mathrm{Min}}\ E_{z,t}\left\{ \int_t^{t+\Delta} L(t,x,u)dt + V\left[(x+\Delta x)_{t+\Delta}, t+\Delta\right]\right\} \tag{B2}$$

where $\bar{x} = E_{z,t}\{x\}$; $\Delta x(t+\sigma)$, $-\tau < \sigma \leq 0$ is a smooth displacement about some given $x(t)$ corresponding to a fixed control u; and the conditional expectation operation is performed on x under the condition that Δx is determinate.

It now follows that for some sample functions of the x-process

$$V\left[(x+\Delta x)_{t+\Delta}, t+\Delta\right] = V\left[x_t, t\right] + \frac{dV}{dt}\Delta + \frac{1}{2}\frac{d^2V}{dt^2}\Delta^2 + o(\Delta^3) \tag{B3}$$

Here the third partial derivatives of V[...] are assumed to be bounded; dV/dt is the total derivative of V[...] with respect to t evaluated

along a sample trajectory $x(t)$ for some fixed control; similarly, d^2V/dt^2; $\lim o(\Delta^3)/\Delta^2 = C$ as Δ approaches to zero. The expectation (the ensemble average) is to be taken over all x-trajectories.

Equations (B3) is now substituted into the right side of equation (B2). Since $V[\bar{x}_t, t]$ does not depend explicitly on $u(\cdot)$, it can be taken outside the braces in the resulting equation; it then cancels the term on the left side. One thus obtains

$$0 = \underset{u}{\text{Min}}\, E_{z,t}\left\{\Delta L[t,x,u] + \Delta \frac{dV}{dt} + \frac{\Delta^2}{2} \frac{d^2V}{dt^2} + o(\Delta^3)\right\} \tag{B4}$$

In order to demonstrate the implication of equation (B4) and of the functional dependence of V upon x, the problem is considered in which the plant is linear with additive noise terms and the ensemble average of the performance criterion is quadratic in control and state variables. In such a case, the functional expression of $V[\cdot,\cdot]$ can be written as

$$\begin{aligned} V[\tilde{x}_t, t] = E_{z,t} \Big\{ \|x(t)\|^2_{P_0(t)} &+ x'(t)\int_{-\tau}^{0} P_1(t,\sigma)x(t+\sigma)d\sigma + \int_{-\tau}^{0} x'(t+\sigma)\,\cdot \\ &P_1'(t,\sigma)d\sigma\, x(t) + \int_{-\tau}^{0}\int_{-\tau}^{0} x(t+\sigma)P_2(t,\sigma,\alpha)x(t+\alpha)d\sigma\, d\alpha + S(t)\Big\} \end{aligned} \tag{B5}$$

where $P_0(t)$, $P_1(t,\sigma)$ and $P_2(t,\sigma,\alpha)$ are matrices to be determined; $S(t)$ is a scalar. Now, the expression of dV/dt for a sample function and some deterministic $\Delta x(t)$ can be written

$$\frac{dV}{dt}\Delta = \frac{\partial V}{\partial t}\Delta + \Delta x'(t)\, P_o(t)x(t) + x'(t)P_o(t)\Delta x(t) + \Delta x'(t)\int_{-\tau}^{0} P_1(t,\sigma)x(t+\sigma)d\sigma$$

$$+ x'(t)\int_{-\tau}^{0} P_1(t,\sigma)\Delta x(t+\sigma)d\sigma + \int_{-\tau}^{0} \Delta x'(t+\sigma)P_1'(t,\sigma)d\sigma\, x(t) +$$

$$+ \int_{-\tau}^{0} x'(t+\sigma)P_1'(t,\sigma)d\sigma\, \Delta x(t) + \int_{-\tau}^{0}\int_{-\tau}^{0}\Big[\Delta x'(t+\sigma)P_2(t,\sigma,\alpha)x(t+\alpha)$$

$$+ x(t+\sigma)P_2(t,\sigma,\alpha)\Delta x(t+\alpha)\Big]d\sigma\, d\alpha + \dot{S}(t)\,\Delta\Big\} \quad (B6)^*$$

Similarly, d^2V/dt^2 can be obtained.

It is now observed that $\Delta x(t)$ is actually a random variable. In order to determine the expression in the braces of equation (B4), both $\Delta\, dV/dt$ and $\Delta^2\, d^2V/dt^2$ must be averaged (ensemble) over all possible $\Delta x(t+\xi)$, $-\tau < \xi \leq 0$. Now, $dw(t)$ and $\Delta x(t)$ are Gaussian processes. Moreover, $\Delta x(t)$ evolves in time according to equation (B7):

$$\Delta x(t) = \Big[A_1 x(t) + A_2 x(t-\tau) + Bu(t)\Big]\Delta + D(t)\, dw(t) \quad (B7)$$

The average value of $\Delta x(t)$ for a given sample function $x(t)$ evolves in time according to

$$\overline{\Delta x}(t) = \Big[A_1\, x(t) + A_2\, x(t-\tau) + Bu(t)\Big]\Delta \quad (B8)$$

since $E_{x,t}\{\cdot\} = E_{x,t}\{E[\cdot\,|x(t)]\}$ where E signifies the ensemble average with respect to Δx.

It follows that the probability density functions $p(\cdot)$ of dW and Δx are

$$p(dw(t)) = \text{const}\,\exp\Big[-\tfrac{1}{2}\,(dw)'(Q_1\Delta)^{-1}(dw)\Big] \quad (B9)$$

---

*In this appendix, $\Delta x(t)$ is written for $dx(t)$.

$$p\left[\Delta x(t+\sigma)\,|\,x(t), x(t+\sigma), t,\ -\tau < \sigma \leq 0\right] = \tag{B10}$$

$$= \text{const.}\ \exp\left\{-\frac{1}{2}\left[\Delta x(t+\sigma) - \overline{\Delta x}(t+\sigma)\right]'\Sigma^{-1}/\Delta\left[\Delta x(t+\sigma) - \overline{\Delta x}(t+\sigma)\right]\right\}$$

where $\|\cdot\|^2_{\Sigma^{-1}} = (\cdot)'\Sigma^{-1}(\cdot)$; and $\Sigma = D(t+\sigma)Q_1D'(t+\sigma)$.

Since the probability density functions of dw and Δx are known, the ensemble average (with respect to Δx) of the expression in equation (B4) can be evaluated by means of equations (B5) through (B11):

$$E_{z,t}\left\{\Delta L(t,x,u) + \Delta\frac{\partial V(x_t,t)}{\partial t} + E_{z,t}\left[A_1x(t) + A_2x(t-\tau) + Bu(t)\right]\Delta\cdot\right.$$

$$\left[P_o(t)x(t) + \int_{-\tau}^{0}P_1(t,\sigma)x(t+\sigma)d\sigma\right] + \left[x'(t)P_o(t) + \int_{-\tau}^{0}x'(t+\sigma)P_1'(t,\sigma)d\sigma\right]\cdot$$

$$E_{z,t}\left[A_1x(t)+A_2x(t-\tau) + Bu(t)\right]\Delta + \int_{-\tau}^{0}\left[x'(t)P_1(t,\sigma)\overline{\Delta x}(t+\sigma) +\right.$$

$$\left.\overline{\Delta x}'(t+\sigma)P_1'(t,\sigma)x(t)\right]d\sigma + \int_{-\tau}^{0}\int_{-\tau}^{0}\left[\overline{\Delta x}'(t+\sigma)P_2(t,\sigma,\alpha)x(t+\alpha) + x'(t+\sigma)\cdot\right.$$

$$\left.P_2(t,\sigma,\alpha)\overline{\Delta x}(t+\alpha)\right]d\sigma\,d\alpha + \frac{\Delta}{2}\,\text{tr}\left[P_o(t)DQ_1D'(t) + \frac{1}{2}P_1(t,0)DQ_1D'(t) +\right.$$

$$\left.\left.\frac{1}{2}P_1'(t,0)DQ_1D'(t) + \int_{-\tau}^{0}P_2(t,\sigma,\sigma)D(t+\sigma)Q_1D'(t+\sigma)d\sigma\right] + \Delta\dot{S}(t) + o(\Delta^2)\right\} \tag{B11}$$

Expression (B11) can now be substituted into equation (B4). Dividing through by Δ in the resulting equation and letting Δ approach to zero, one obtains:

$$0 = \underset{u}{\text{Min}}\ E_{z,t}\ \Big\{L(t,x,u) + \frac{\partial V}{\partial t} + \Big[A_1x(t) + A_2x(t-\tau) + Bu(t)\Big]'\Big[P_o(t)x(t) +$$

$$\int_{-\tau}^{0} P_1(t,\sigma)x(t+\sigma)d\sigma\Big] + \Big[x'(t)P_o(t) + \int_{-\tau}^{0} x'(t+\sigma)P_1'(t,\sigma)d\sigma\Big]\Big[A_1x(t) + A_2x(t-\tau) +$$

$$Bu(t)\Big] + \lim_{\Delta\to 0}\int_{-\tau}^{0}\Big[x'(t)P_1(t,\sigma)\ \frac{\overline{\Delta x}(t+\sigma)}{\Delta} + \frac{\overline{\Delta x}(t+\sigma)}{\Delta}\ P_1'(t,\sigma)x(t)\Big]\,d\sigma +$$

$$+ \lim_{\Delta\to 0}\int_{-\tau}^{0}\int_{-\tau}^{0}\Big[\frac{\overline{\Delta x}'(t+\sigma)}{\Delta}\ P_2(t,\sigma,\alpha)x(t+\alpha) + x'(t+\sigma)P_2(t,\sigma,\alpha)\ \frac{\overline{\Delta x}(t+\alpha)}{\Delta}\Big]\,d\sigma\ d\alpha +$$

$$+ \dot{S}(t) + \frac{1}{4}\ \mathrm{tr}\Big[\Big(2P_o(t) + P_1(t,0) + P_1'(t,0)\Big)\ DQ_1D'(t) + 2\int_{-\tau}^{0} P_2(t,\sigma,\sigma)DQ_1D'(t+\sigma)d\sigma\Big]\Big\} \tag{B12}$$

It is noted that in expression (B12), the term $\overline{\Delta x}(\cdot)$ is given by equation (B8). Since $E[\Delta x|z(t)] = E[E\ [\Delta x|z(t), x(t),\ t_o \leq t]]$, it follows that $\lim_{\Delta\to 0} E[\overline{\Delta x}|z(t), x(t),\ t_o \leq t]/\Delta = d\bar{x}(\cdot)/dt$ exists for all sample functions of the $x(\cdot)$ process.

Since $\partial\bar{x}(t+\sigma)/\partial t = \partial\bar{x}(t+\sigma)/\partial\sigma$, an integration by parts in equation (B13) leads to

$$0 = \underset{u}{\text{Min}}\ E_{z,t}\ \Big\{x'(t)Wx(t) + u'(t)Ru(t) + \frac{\partial V}{\partial t} + \Big[A_1x(t) + A_2x(t-\tau) + Bu(t)\Big]'\ \cdot$$

$$\Big[P_o(t)x(t) + \int_{-\tau}^{0} P_1(t,\sigma)x(t+\sigma)d\sigma\Big] + \Big[x'(t)P_o(t) + \int_{-\tau}^{0} x'(t+\sigma)P_1'(t,\sigma)d\sigma\Big]\ \cdot$$

$$\Big[A_1x(t) + A_2x(t-\tau) + Bu(t)\Big] + \int_{-\tau}^{0}\Big[x'(t)\ \frac{\partial P_1(t,\sigma)}{\partial\sigma}\ x(t+\sigma) + x'(t+\sigma)\ \frac{\partial P_1'(t,\sigma)}{\partial\sigma}\ x(t)\Big]d\sigma +$$

$$\int_{-\tau}^{0}\int_{-\tau}^{0}\Big[x'(t+\sigma)\ \frac{\partial P_2(t,\sigma,\alpha)}{\partial\sigma}\ x(t+\alpha) + x'(t+\sigma)\ \frac{\partial P_2(t,\sigma,\alpha)}{\partial\alpha}\ x(t+\alpha)\Big]\,d\sigma\ d\alpha +$$

$$+ x'(t)P_1(t,0)x(t) + x'(t)\dot{P}_1'(t,0)x(t) - x'(t)P_1(t,-\tau)x(t-\tau) -$$

$$x'(t-\tau)P_1'(t,-\tau)x(t) + \int_{-\tau}^{0}\left[x'(t)\dot{P}_2(t,0,\alpha)x(t+\alpha) - x'(t-\tau)P_2(t,-\tau,\alpha)x(t+\alpha)\right]d\alpha$$

$$+ \int_{-\tau}^{0}\left[x'(t+\sigma)P_2(t,\sigma,0)x(t) - x'(t+\sigma)P_2(t,\sigma,-\tau)x(t-\tau)\right]d\sigma +$$

$$\dot{S}(t) + \frac{1}{4}\mathrm{tr}\left[\left(2P_0(t)+P_1(t,0)+P_1'(t,0)\right)DQ_1D'(t) + 2\int_{-\tau}^{0} P_2(t,\sigma,\sigma)DQ_1D'(t+\sigma)d\sigma\right]\Bigg\}$$

(B13)

In equation (B13), the terms $\dot{S}(t)$ and tr[·] do not depend upon the control. Hence, the optimal control can be obtained from equation (B13) by expressing the control u as a complete square. When the optimal control u^* is then substituted back into equation (B12), equations for determining $P_0(t)$, $P_1(t,\sigma)$ and $P_2(t,\sigma,\alpha)$ are obtained.

PART II

On Certainty-Equivalence and Certainty-Difference In Stochastic Linear Systems with Time Delay

ON CERTAINTY-EQUIVALENCE AND CERTAINTY-DIFFERENCE IN STOCHASTIC LINEAR SYSTEMS WITH TIME DELAY

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The optimization of stochastic linear systems with time delay is presented in the framework of the dynamic programming method. A sufficient condition for the optimality is obtained by applying the principle of optimality. An example is presented to demonstrate that the certainty-equivalence is valid in optimizing a class of stochastic linear systems with time delay. Its validity, however, is quite restricted. Another example in which the variance of the additive noise in the differential-difference equation depends upon the control illustrates that a certainty-difference may be as well encountered.

Introduction

It is feasible that the optimal solution of a stochastic dynamical system agrees with that of the deterministic dynamical system obtained by formally replacing the random variables in the stochastic system by their expected values. Such a coincidence is usually called certainty equivalence [1,2]. It is known to hold, for example, if (i) the plant dynamics is described by a stochastic linear (ordinary) differential equation containing an

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additive Gaussian noise; (ii) the covariance of the plant noise is independent of the control and state variables; and (iii) the expected value of performance criterion to be minimized is a quadratic functional in control and states. However, the certainty equivalence is not valid in stochastic linear (ordinary) differential equations if the covariance of the plant noise depends upon the control. In fact, in such a case, a certainty-difference is encountered [1].

The purpose here is first to present the optimization of stochastic linear systems with time delay in the framework of the dynamic programming method. Then, it is demonstrated that the "principle" of certainty-equivalence as well as the "principle" of certainty-difference are encountered also in optimizing linear stochastic systems with time delay.

Problem Statement

The state transition of a system is governed by a stochastic scalar (for convenience) differential-difference equation

$$dx(t) = [a_1(t)x(t) + a_2(t)x(t-\tau) + b(t)u(t)]dt + a_3(t)d\xi(t) \quad (1)$$

where $t \in [t_o, T]$, $T < \infty$; $x(t)$ represents the system state at time t and $x(t-\tau)$ at time $t-\tau$ (time-lag τ = const); $a_1(t), a_2(t), a_3(t)$ and $b(t)$ are bounded C'-functions of t; $\xi(t)$ is a Brownian motion process.

The observation equation is

$$dz(t) = [c_1(t)x(t) + c_2(t)x(t-\tau)]dt + d\eta(t) \quad (2)$$

where $c_1(t)$ and $c_2(t)$ are bounded and continuous in t and $\eta(t)$ is a Brownian motion process.

The initial function for equation (1) is determined by a Gaussian process specified by

$$E[x(t_o+\sigma)] = \bar{x}(t_o+\sigma), \quad -\tau < \sigma, \ \theta \leq 0 \tag{3}$$

$$E[(x(t_o+\sigma) - \bar{x}(t_o+\sigma)) \ (x(t_o+\theta) - \bar{x}(t_o+\theta))] = C(t_o, \sigma, \theta) \tag{4}$$

The $\xi(t)$ and $\eta(t)$ processes are zero-mean processes with variances, respectively,

$$E[\xi(t_1)\xi(t_2)] = Q_1(t_1)\delta(t_1-t_2) \tag{5}$$

$$E[\eta(t_1)\eta(t_2)] = Q_2(t_1)\delta(t_1-t_2) \tag{6}$$

where $Q_1(\cdot)$ and $Q_2(\cdot)$ are given.

It is also assumed that $x(t)$, $t \in [t_o-\tau, t_o)$, $\xi(t)$, and $\eta(t)$, $t \in [t_o, T]$ represent independent random processes.

The problem is to determine the deterministic optimal control $u^*(t, \bar{x})$ so as to minimize

$$J[u] = E_{z,t} \left\{ \int_{t_o}^{T} [w(\sigma)x^2(\sigma) + r(\sigma)u^2(\sigma)]d\sigma \right\} \tag{7}$$

where T is given, $w(\sigma)$ and $r(\sigma)$ are positive for all σ, and $E_{z,t}\{\cdot\} = E\{\cdot \mid z(\tau), \ t_o \leq \tau \leq t\}$ signifies the conditional expectation.

The optimal control problem posed is solved by the application of Bellman's principle of optimality. Sufficient equations for the optimality are first presented. Then, these equations are applied to two examples. The one demonstrates a case in which the "principle" of certainty-equivalence is valid; the other illustrates certainty-difference in stochastic systems with time delay. For the former case, it is shown that

the resulting optimal control is the same as the one obtained in the deterministic problem by minimizing

$$\bar{J}[u] = \int_{t_o}^{T} [w(\sigma)\bar{x}^2(\sigma) + r(\sigma)u^2(\sigma)]\, d\sigma \tag{8}$$

subject to the constraint

$$\dot{\bar{x}}(t) = a_1(t)\bar{x}(t) + a_2(t)\bar{x}(t-d) + b(t)u(t),\ t \in [t_o, T] \tag{9}$$

where $\bar{x}(t) = E_{z,t}\{x(t)\}$. Equations (8) and (9) have been obtained by substituting the mean values in equation (1) and (7) for the random variables. For the latter case, it is shown that, if the variance $Q_1(.)$ in equation (5) depends upon the control $u(.)$, the optimal control that minimizes (7) subject to the constraint equation (1) is not the same as the optimal control that minimizes (8) subject to the constraint equation (9).

Preliminary Material

Some introductory material on the linear differential-difference equation is first presented [2,3]. When the state of the system is governed by the differential-difference equation (1), the evolution of the process $x(t)$, $t \geq t_o$ is defined by

$$x(t) = \phi(t,t_o)x(t_o) + \int_{-\tau}^{0} \phi(t,t_o+\tau+\alpha)a_2(t_o+\tau+\alpha)x(t_o+\alpha)d\alpha$$

$$+ \int_{t_o}^{t} \phi(t,\alpha)b(\alpha)u(\alpha)d\alpha + \int_{t_o}^{t} \phi(t,\alpha)a_3(\alpha)d\xi(\alpha) \tag{10}$$

where $\phi(t,t_o)$, $t \geq t_o$ is the solution to the homogenous part $[u(t) = 0]$

of equation (9) with the boundary condition $\phi(t,t) = 1$ and $\phi(t,s) = 0$, if $t < s$.

If $u(.)$ is assumed to be linear in the state variable, then equation (10) can be used to demonstrate that the functional in equation (7) is of the form

$$J[u] = P_o(t_o)\bar{x}^2(t_o) + 2\bar{x}(t_o)\int_{-\tau}^{0} P_1(t_o,\sigma)\bar{x}(t_o+\sigma)d\sigma + \int_{-\tau}^{0}\int_{-\tau}^{0} \bar{x}(t_o+\sigma)P_2(t_o,\sigma,\alpha)\bar{x}(t_o+\alpha)d\sigma d\alpha + S(t_o) \tag{11}$$

where $P_o(t_o)$, $P_1(t_o,\sigma)$ and $P_2(t_o,\sigma,\alpha)$ and $S(t_o)$ are independent of x. The form of equation (11) will be used in the sequel to determine the optimal solution. Also, the equations specifying $P_o(t)$, $P_1(t,\sigma)$, and $P_2(t,\sigma,\alpha)$ will then be obtained.

Solution by Means of Dynamic Programming

To determine the solution to the stochastic optimum control problem, the principle of optimality is applied. When the system starts at time t from state $x(t+\sigma)$, $-\tau < \sigma \leq 0$, the minimum value of the return-function is denoted by

$$V[\bar{x}_t, t] = \underset{u}{\text{Min}}\ E_{z,t}\left\{\int_t^T [w(\sigma)x^2(\sigma) + r(\sigma)u^2(\sigma)]d\sigma\right\} \tag{12}$$

By the principle of optimality, one obtains

$$V[\bar{x}_t, t] = \underset{u}{\text{Min}}\ E_{z,t}\left\{\int_t^{t+\Delta} [wx^2(\sigma) + ru^2(\sigma)]d\sigma + V[\overline{x(t+\Delta)}_{t+\Delta}, t+\Delta]\right\} \tag{13}$$

where Δ represents a small time-increment. The expression in the braces

on the right of equation (13) is expanded about the function representing the evolution of the mean value $\bar{x}$ of the x-process. The term $V[\bar{x}_t, t]$ appears now on both sides of the resulting equation. Since it is independent upon the control, it can be canceled. By assuming that the third partial derivatives of $V[.,.]$ are bounded, and dividing through by Δ, one obtains

$$0 = \underset{u}{\text{Min}}\ E_{z,t} \left\{ wx^2(t) + ru^2(t) + \left(\frac{dV}{dt}\right)_{\bar{x}(t)} + \frac{1}{2}\left(\frac{d^2V}{dt^2}\right)_{\bar{x}(t)} \Delta + o(\Delta) \right\} \qquad (14)$$

where $\lim o(\Delta)/\Delta = 0$ as Δ approaches zero. Equation (14) is the basic equation to be used in solving the stochastic control problem.

In the specific case where the plant is linear and the return function quadratic in state variables and control, $V[\bar{x}_t, t]$ is assumed to have the form presented in equation (11), where t_o is replaced by t, and $P_o(t)$, $P_1(t,\sigma)$, and $P_2(t,\sigma,\alpha)$ are matrices to be determined. This expression of $V[.,.]$ is now substituted into equation (14). Forming dV/dt and d^2V/dt^2 about the average trajectory $\bar{x}(t)$, substituting $dx(t)$ from equation (1), performing appropriate expectations in the resulting expressions relative first to dx (or $d\xi$) and then to x given $z(t)$, $t_o \leq t$, one obtains

$$\begin{aligned} 0 = \underset{u}{\text{Min}} \Big\{ & w\bar{x}^2(t) + wC(t,0,0) + ru^2(t) + \frac{\partial V}{\partial t} + 2[a_1\bar{x}(t) + a_2\bar{x}(t-\tau) + bu] \\ & [P_o(t)\bar{x}(t) + \int_{-\tau}^{0} P_1(t,\sigma)\bar{x}(t+\sigma)d\sigma] + 2\int_{-\tau}^{0} \bar{x}(t)P_1(t,\sigma)\frac{\partial \bar{x}(t+\sigma)}{\partial \sigma}d\sigma + \\ & + \int_{-\tau}^{0} d\sigma \int_{-\tau}^{0} d\alpha \left[\frac{\partial \bar{x}(t+\sigma)}{\partial \sigma} P_2(t,\sigma,\alpha)\bar{x}(t+\alpha) + \bar{x}(t+\sigma)P_2(t,\sigma,\alpha)\frac{\partial \bar{x}(t+\alpha)}{\partial \alpha} \right] \\ & + \frac{1}{2}[P_o(t) + P_1(t,0)]a_3^2(t)Q_1 + \frac{1}{2}\int_{-\tau}^{0} P_2(t,\sigma,\sigma)a_3^2(t+\sigma)Q_1 d\sigma \Big\} \end{aligned} \qquad (15)$$

Hence, the optimal control is determined by minimizing the deterministic expression in the braces of equation (15). The solutions to two different optimum control problems are now given; the details can be found in the appendix. In the first one, the variance Q_1 of the plant noise is constant; in the second one, the same variance is dependent upon the control.

Certainty-Equivalence

Suppose that the variance Q_1 of the plant noise is constant. Equation (15) determines now the optimal control:

$$u^*(t,\bar{x}_t) = -\frac{1}{r} b(t)[P_o(t)\bar{x}(t) + \int_{-\tau}^{0} P_1(t,\sigma)\bar{x}(t+\sigma)d\sigma] \qquad (16)$$

Hence, the optimal control is linear in the estimated value $\bar{x}$ of the state variable. Substituting equation (16) back into equation (15), equations for determining $P_o(t)$, $P_1(t,\sigma)$, and $P_2(t,\sigma,\alpha)$ result.

$$\frac{dP_o(t)}{dt} + 2a_1(t)P_o(t) + 2P_1(t,0) - \frac{b^2(t)}{r(t)} P_o^2(t) + w(t) = 0 \qquad (17)$$

$$\frac{\partial P_1(t,\sigma)}{\partial t} - \frac{\partial P_1(t,\sigma)}{\partial\sigma} + a_1(t)P_1(t,\sigma) - \frac{b^2(t)}{r(t)} P_o(t)P_1(t,\sigma) + P_2(t,0,\sigma) = 0 \qquad (18)$$

$$\frac{\partial P_2(t,\sigma,\alpha)}{\partial t} - \frac{\partial P_2(t,\sigma,\alpha)}{\partial\sigma} - \frac{\partial P_2(t,\sigma,\alpha)}{\partial\alpha} - \frac{b^2(t)}{r(t)} P_1(t,\sigma)P_1(t,\alpha) = 0 \qquad (19)$$

$$\dot{S}(t) + w(t)C(t,0,0) + \frac{1}{2} [P_o(t)+P_1(t,0)]a_3^2(t)Q_1 + \frac{1}{2}\int_{-\tau}^{0} P_2(t,\sigma,\sigma)\cdot a_3^2(t+\sigma)Q_1 d\sigma = 0 \qquad (20)$$

$$P_o(T) = \underline{0}\ ; \quad P_1(T,\sigma) = \underline{0}\ ; \quad P_2(T,\sigma,\alpha) = \underline{0}\ ; \quad S(T) = 0\ ; \quad -\tau < \sigma \leq 0$$

$$a_2(t)P_o(t) - P_1(t,-\tau) = 0 \qquad\qquad ; \quad -\tau < \alpha \leq 0 \qquad (21)$$

$$a_2(t)P_1(t,\sigma) - P_2(t,-\tau,\sigma) = 0$$

It is emphasized that the realization of the optimal control requires the determination of the estimates $\hat{x}(t+\sigma) = E_{z,t}[x(t+\sigma)]$, $-\tau < \sigma \leq 0$ of the state variables. The expected values of the state variables conditioned on the measurements furnish the minimum for the average value of the integrated estimation error squared. They can be obtained by the application of the results on the optimal filtering in linear stochastic systems with time delay [6]. The feedback gains in equation (16) can be computed "off-line" before the application of the optimal control. Moreover, they are not functions of the observations $z(t)$ or of the state variables $x(t)$. Consequently, the determination of the optimal control is independent of the optimal estimation. This statement is known as a "separation theorem" in conjunction with stochastic optimum control problems.

If the optimum control problem described by system (8) and (9) obtained by substituting only mean values for the random variables is solved , the resulting optimal control is exactly the same as the one given in equation (16). In fact, the feedback gains are also specified by the same equations ((17) through (21)). This coincidence is usually termed the "principle" of certainty equivalence, which is here established for a class of stochastic systems with time delay.

Certainty-Difference

Suppose the variance Q_1 of the plant noise is dependent upon the control, say, $Q_1=2q_1u^2$, where q_1 is a constant. This situation occurs, for example, if the randomness in the plant is produced by the application of the control. Hence, the random variable in the plant equation is zero if the control is zero.

The optimal control is again determined by equation (15). If $Q_1=2q_1u^2$ is substituted into equation (15), the minimization operation yields for the optimal control

$$u^*(t,\bar{x}_t) = -b(t)\ \frac{P_o(t)\bar{x}(t) + \int_{-\tau}^{0} P_1(t,\alpha)\bar{x}(t+\alpha)d\alpha}{r + [P_o(t)+P_1(t,0)]a_3^2(t)q_1}\ ;\ \sigma=0$$

$$u^*(t+\sigma,\bar{x}_t) = 0, \quad -\tau < \sigma < 0 \tag{22}$$

where $\bar{x}(t+\sigma) = E_{z,t}\{x(t+\sigma)\}$. It is noted that r is positive definite; $P_o(t)$ and $P_1(t,0)$ are nonnegative for $t \in [t_o, T]$; and $P_2(t,\sigma,\sigma)$ is nonnegative [5] in the domain of its definition. Substituting equation (22) back into equation (15), equations for determining $P_o(t)$, $P_1(t,\sigma)$ and $P_2(t,\sigma,\alpha)$ result.

$$\frac{dP_o(t)}{dt} + 2a_1(t)P_o(t)+2P_1(t,0) + w - \frac{b^2(t)P_o^2(t)}{r_a(t)}\left[2 - \frac{r(t)}{r_a(t)}\right] = 0 \tag{23}$$

where $r_a(t) = r(t)+[P_o(t)+P_1(t,0)]\ a_3^2(t)q_1$;

$$\frac{\partial P_1(t,\sigma)}{\partial t} - \frac{\partial P_1(t,\sigma)}{\partial \sigma} + a_1(t)P_1(t,\sigma)+P_2(t,0,\sigma) - \frac{b^2(t)P_o(t)P_1(t,\sigma)}{r_a(t)}\left[2-\frac{r(t)}{r_a(t)}\right] = 0 \tag{24}$$

$$\frac{\partial P_2(t,\sigma,\alpha)}{\partial t} - \frac{\partial P_2(t,\sigma,\alpha)}{\partial \sigma} - \frac{\partial P_2(t,\sigma,\alpha)}{\partial \alpha} - \frac{b^2(t)P_1(t,\alpha)P_1(t,\sigma)}{r_a(t)}\left[2-\frac{r(t)}{r_a(t)}\right] = 0 \tag{25}$$

$$\frac{dS(t)}{dt} + w(t)\ C(t,0,0) = 0 \tag{26}$$

The boundary conditions are the same as those given in equation (21).

Suppose now that the optimal control is determined on the basis of the certainty-equivalence. The resulting control is given by equation (16). It does not agree with the optimal control specified by equation (22). Indeed, the use of the certainty-equivalence leads to an incorrect answer. The example presented here demonstrates a certainty-difference in stochastic linear systems with time delay.

Conclusions

A sufficient condition for the optimization of stochastic linear systems with time delay is formally derived by means of the dynamic programming method. The result is then applied to a system in which the plant described by differential-difference equations is linear containing additive Gaussian noise. It is demonstrated that the "principle" of certainty-equivalence holds when the variance of the plant noise is constant. However, the use of the certainty-equivalence leads to an incorrect answer when the variance of the plant noise depends upon the control. In fact, a certainty-difference is encountered in such an example.

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Appendix

Formal Derivation of Equations(15), (16), (22)

By the application of the principle of optimality, equation (12) can be expressed in the form given by equations (13) and (14). The derivatives dV/dt and d^2V/dt^2 in equation (14) are evaluated about the trajectory $\bar{x}(t)$. It is stipulated that

$$V[\bar{x}_t, t] = P_o(t)\bar{x}^2(t) + 2\bar{x}(t)\int_{-\tau}^{0} P_1(t,\sigma)\bar{x}(t+\sigma) +$$

$$+ \int_{-\tau}^{0} d\sigma \int_{-\tau}^{0} d\alpha \left[\bar{x}(t+\sigma)P_2(t,\sigma,\alpha)\bar{x}(t+\alpha)\right] + S(t) \qquad \text{(A1)}$$

It follows that

$$\left(\frac{dV}{dt}\right)_{\bar{x}(t)} \Delta = \left(\frac{\partial V}{\partial t}\right)_{\bar{x}(t)} \Delta + 2(dx(t))\left[P_o(t)\bar{x}(t) + \int_{-\tau}^{0} P_1(t,\sigma)\bar{x}(t+\sigma)d\sigma\right] +$$

$$+ 2\bar{x}(t)\int_{-\tau}^{0} P_1(t,\sigma)\ (dx(t+\sigma))\ d\sigma + \qquad \text{(A2)}$$

$$+ \int_{-\tau}^{0} d\sigma \int_{-\tau}^{0} d\alpha \left[(dx(t+\sigma))P_2(t,\sigma,\alpha)\bar{x}(t+\alpha) + \bar{x}(t+\sigma)P_2(t,\sigma,\alpha)(dx(t+\alpha))\right]$$

where $dx(t)$ is considered as determinate. Similarly, d^2V/dt^2 can be written. In expression (A2), $dx(t)$ is actually a random variable, which evolves in time according to equation (1). The expression (A2) is then substituted for $(dV/dt)_{\bar{x}(t)}$ in equation (14). The term $(d^2V/dt^2)_{\bar{x}(t)}$ in equation (14) is replaced in the same manner. The resulting expression can be written as follows:

$$0 = \underset{u}{\text{Min}}\, E_{z,t} \Big\{wx^2 + ru^2 + \left(\frac{\partial V}{\partial t}\right)_{\bar{x}(t)} + 2(dx(t))\Big[P_0\bar{x}(t) + \int_{-\tau}^{0} P_1(t,\sigma)\bar{x}(t+\sigma)d\sigma\Big]\Big/\Delta +$$

$$+ 2\bar{x}(t)\int_{-\tau}^{0} P_1(t,\sigma)\ (dx(t+\sigma))d\sigma/\Delta +$$

$$+ \int_{-\tau}^{0} d\sigma \int_{-\tau}^{0} d\alpha\Big[(dx(t+\sigma))P_2(t,\sigma,\alpha)\bar{x}(t+\alpha) + \bar{x}(t+\sigma)P_2(t,\sigma,\alpha)(dx(t+\alpha))\Big]\Big/\Delta +$$

$$+ \frac{1}{2}\left(\frac{d^2V}{dt^2}\right)_{\bar{x}(t)} \Delta + o(\Delta^2)\Big\} \qquad \text{(A3)}$$

In order to perform the expectation operation, the probability density functions associated with $d\xi$ and dx are written:

$$p[d\xi(t)] = \text{const.}\ \exp[-\tfrac{1}{2}(d\xi)^2/(Q_1\Delta)] \qquad \text{(A4)}$$

$$p[dx(t+\sigma)\,|\,x(t), z(t), t, -\tau < \sigma \leq 0] =$$

$$\text{const.}\ \exp\Big\{-\tfrac{1}{2}\,[dx(t+\sigma) - \overline{dx}(t+\sigma)]^2/[\Delta\ a_3^2(t+\sigma)Q_1] \qquad \text{(A5)}$$

where $\overline{dx}(t)$ is governed by equation

$$\overline{dx}(t) = [a_1x(t) + a_2x(t-\tau) + bu(t)]\ \Delta \qquad \text{(A6)}$$

Now, $dx(t)$ from equation (1) is substituted into equation (A3). Since

$$E[\Delta x\,|\,z(t), t, t_o \leq t] = EE[\Delta x\,|\,x(t), z(t), t, t_o \leq t] = E_{z,t}E[\Delta x\,|\,x(t), t_o \leq t],$$

one can perform the averaging in equation (A3) (after the substitution of dx) and the limiting process as Δ approaches to zero:

$$0 = \underset{u}{\mathrm{Min}} \left\{ w\bar{x}^2(t) + wC(t,0,0) + ru^2(t) + \left(\frac{\partial V}{\partial t}\right)_{\bar{x}(t)} + 2\bar{x}(t) \int_{-\tau}^{0} P_1(t,\sigma) \frac{\partial \bar{x}(t+\sigma)}{\partial \sigma} d\sigma + \right.$$

$$+ 2\left[a_1\bar{x}(t) + a_2\bar{x}(t-\tau) + bu(t)\right]\left[P_0\bar{x}(t) + \int_{-\tau}^{0} P_1(t,\sigma)\bar{x}(t+\sigma)d\sigma\right] +$$

$$+ \int_{-\tau}^{0} d\sigma \int_{-\tau}^{0} d\alpha \left[\frac{\partial \bar{x}(t+\sigma)}{\partial \sigma} P_2(t,\sigma,\alpha)\bar{x}(t+\alpha) + \bar{x}(t+\sigma)P_2(t,\sigma,\alpha) \frac{\partial \bar{x}(t+\alpha)}{\partial \alpha}\right] +$$

$$\left. + \frac{1}{2}\left[P_0(t) + P_1(t,0)\right] a_3^2(t)Q_1 + \frac{1}{2}\int_{-\tau}^{0} P_2(t,\sigma,\sigma)a_3^2(t+\sigma)Q_1 d\sigma \right\} \qquad \text{(A7)}$$

where use has been made of the fact that $\lim_{\Delta \to 0} E[\Delta x(t)/\Delta | x(t),\ t_0 \leq t] = \lim_{\Delta \to 0} \overline{\Delta x(t)}/\Delta = d\bar{x}(t)/dt$, and that $\partial \bar{x}(t+\sigma)/\partial t = \partial \bar{x}(t+\sigma)/\partial \sigma$.

Suppose now that Q_1 is constant. Since $C(t,0,0)$ and Q_1 do not depend upon the control explicitly, and the terms $\partial \bar{x}(t+\sigma)/\partial \sigma$ and $\partial \bar{x}(t+\alpha)/\partial \alpha$ can be integrated by parts and do not contain the control u explicitly, the minimum in expression (A7) is attained for the control specified by equation (16).

Suppose now that $Q_1 = 2u^2 q_1$. Collecting now the terms in equation (A7), which contain the control u explicitly, one obtains the expression to be minimized:

$$ru^2(t) + 2bu(t)\left[P_0\bar{x}(t) + \int_{-\tau}^{0} P_1(t,\alpha)\bar{x}(t+\alpha)d\alpha\right] + [P_0 + P_1(t,0)]a_3^2(t)q_1u^2(t) +$$

$$+ \int_{-\tau}^{0} P_2(t,\sigma,\sigma)a_3^2(t+\sigma)q_1u^2(t+\sigma)\, d\sigma \qquad \text{(A8)}$$

The minimum of the expression (A8) is achieved by choosing the optimal control $u^*(t,\bar{x}_t)$ and $u^*(t+\sigma,\bar{x}_t)$, $-\tau < \sigma < 0$ as specified by expressions in equation (22). By substituting the expression of the optimal control back into equation (15), collecting terms, the resulting equation yields the equation governing $P_0(t)$, $P_1(t,\sigma)$ and $P_2(t,\sigma,\alpha)$.

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13. ABSTRACT

For linear stochastic systems with time delay, the optimal control is derived that minimizes the ensemble average a quadratic (in states and control) performance measure. The optimal control obtained is functionally dependent upon the expected values of the state variables conditioned on the measurements. It is shown that the optimal control and estimation can be performed independently; i.e., the separation theorem holds for the class of problems considered. The optimal control is linearly dependent upon the best estimates which minimize the expected value of the estimation error squared. The optimization of stochastic linear systems with time delay is presented in the framework of the dynamic programming method. A sufficient condition for the optimality is obtained by applying the principle of optimality. An example is presented to demonstrate that the certainty-equivalence is valid in optimizing a class of stochastic linear systems with time delay. Its validity, however, is quite restricted. Another example in which the variance of the additive noise in the differential-difference equation depends upon the control illustrates that a certainty-difference may be as well encountered.

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